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ASSESSMENT OF A NEW MODEL FOR HEAT FLOW DURING UNIDIRECTIONAL SOLIDIFICATION OF METALS

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Abstract – This study is an evaluation of an exact mathematical model recently developed by the present authors [1, 2] to treat the generalized solidification problem, subject only to the constraint that the interfacial heat-transfer coefficient be an invariant. The technique involves the mathematical expedient of representing components of the interfacial thermal resistance by virtual layers of solid metal and/or mould and is described as the Virtual Adjunct Method (VAM). It is demonstrated that the kinetic and thermal description predicted by the model reduces to that expected in three simple limiting cases previously subjected to exact analysis. It is also shown that the simple kinetic equations $S = A\sqrt{t} - B$, empirically found to be appropriate in some cases, may be derived from the model under certain boundary conditions. These are investigated and applied to specific casting situations and a rationale developed which appears to explain experimental observations. Finally, the model is examined in generalized cases of mixed thermal control. Comparison is made between the predictions of the model and the results of numerical computations for situations in which the thermal resistances of (a) metal and interface and (b) metal, interface and mould are significant. This is done using measured values for heat-transfer coefficients and comparison is extended to encompass experimental kinetic and thermal data. The performance of the model is shown to be excellent.

NOMENCLATURE

a_s, thermal diffusivity of solid metal, $(=k_s/c_s d_s)$ [m^2/s];
A, constant of the empirical equation (14), $(S = A\sqrt{t} - B)$ [$m/s^{1/2}$];
b_s, heat diffusivity of solid metal, $[=\sqrt{(k_s c_s d_s)}]$ [$J/m^2\ s^{1/2}\ K$];
B, constant of the empirical equation (14), $(S = A\sqrt{t} - B)$ [m];
c_m, specific heat mould material [J/kg K];
c_s, specific heat of solid metal [J/kg K];
d_m, density of mould material [kg/m^3];
d_s, density of solid metal [kg/m^3];
E_0, thickness of 'pre-existing' adjunct to mould in virtual system [m];
h_i, Newtonian heat-transfer coefficient of metal/mould interface [$J/m^2\ s\ K$];
H, latent heat of fusion of metal [J/kg];
k_m, thermal conductivity of mould material [J/m s K];
k_s, thermal conductivity of solid metal [J/m s K];
S, thickness of solidified metal in real system [m];
S', thickness of solidified metal in virtual system [m];
S_0, thickness of 'pre-existing' adjunct to metal in virtual systems [m];
t, time from zero point in real system [s];
T, (absolute) temperature in real and virtual systems [K];
T_f, freezing temperature of metal [K];
T_i, (invariant) temperature of hypothetical plane at metal/mould interface [K];
T_m, temperature at any point in the mould [K];
T_0, initial temperature of mould [K];
T_s, temperature at any point in the solidified metal [K];
V, velocity of liquid/solid interface in real system [m/s];
x, distance from metal/mould interface in real system [m];
x', distance from metal/mould interface in virtual systems [m];
α, first constant of equation (1), $(=1/4\ a_s\phi^2)$ [s/m^2];
β, second constant of equation (1), $(=S_0/2\ a_s\phi^2)$ [s/m];
H^*, dimensionless latent heat of fusion of metal, $H/c_s(T_f - T_0)$;
M, ratio of heat diffusivities of solid metal and mould material, $(k_s c_s d_s/k_m c_m d_m)^{1/2}$;
N, square root of ratio of thermal diffusivities of solid metal and mould material, $(a_s/a_m)^{1/2}$;
S^*, dimensionless thickness of solidification, sh_i/k_s;

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t^*, dimensionless time, $th_i^2/k_s c_s d_s$;
T_m^*, dimensionless temperature at any point in the mould, $(T_m - T_0)/(T_f - T_0)$;
T_s^*, dimensionless temperature at any point in the metal, $(T_s - T_0)/(T_f - T_0)$;
$x^*(x>0)$, dimensionless distance into metal from metal/mould interface, xh_i/k_s;
$x^*(x<0)$, dimensionless distance into mould from metal/mould interface, xh_i/k_m;
ϕ, dimensionless solidification constant, equation (4).

INTRODUCTION

THE DEVELOPMENT of a mathematical model to describe the unidirectional solidification of metals is complicated by a number of factors. One of the most problematical of these is the difficulty of simultaneous treatment of heat flow through metal and/or mould by thermal conduction and across the metal/mould interface by Newtonian heat transfer. The mathematical approaches used to tackle this subject may be very broadly grouped into two classes, according to whether or not the treatment incorporates any mathematical approximations. Techniques in which such approximations are made can be further divided into analytical [3–8] and numerical/graphical [9–13] methods. These may be mathematically versatile but often suffer from lack of generality and/or simplicity.

Mathematically exact treatments, on the other hand, frequently necessitate unrealistic physical assumptions. For example, until recently all exact approaches placed restrictions on the value of the interfacial heat-transfer coefficient [14–18]. A model developed by the present authors [1,2], however, requires only that this value be an invariant during the process. (It may readily be shown that the generalized case of variable h_i is not amenable to exact solution.) In the general form of this model [1] the simultaneous handling of conductive and Newtonian heat transfer is achieved through modelling the interfacial thermal resistances (divided into mould and metal side components) by 'pre-existing' adjuncts of solid. For calculation purposes these are additive (at the interface) to the real physical thickness and heat flow may then be completely described by manipulation of the basic Fourier conduction equations.

It has been shown [1,2] that the Virtual Adjunct Method (VAM) gives a mathematically exact description of the generalized solidification problem, in which both interfacial resistance and thermal capacity of solidified metal are of significance. It describes, of course, an approximation to the real physical situation and requires that interfacial heat transfer be modelled by a constant coefficient.

In this paper, the equations representing the predictions of the model are presented. These are first examined for three simple limiting cases, which correspond to previously-developed exact treatments. It will be demonstrated that the kinetic and thermal behaviour predicted by the model conforms to expected mathematical descriptions in these extrema.

It will then be shown that a classical empirical equation sometimes found appropriate to describe the kinetics of solidification may be derived by manipulation of the model under certain boundary conditions. From this treatment, deductions may be made about

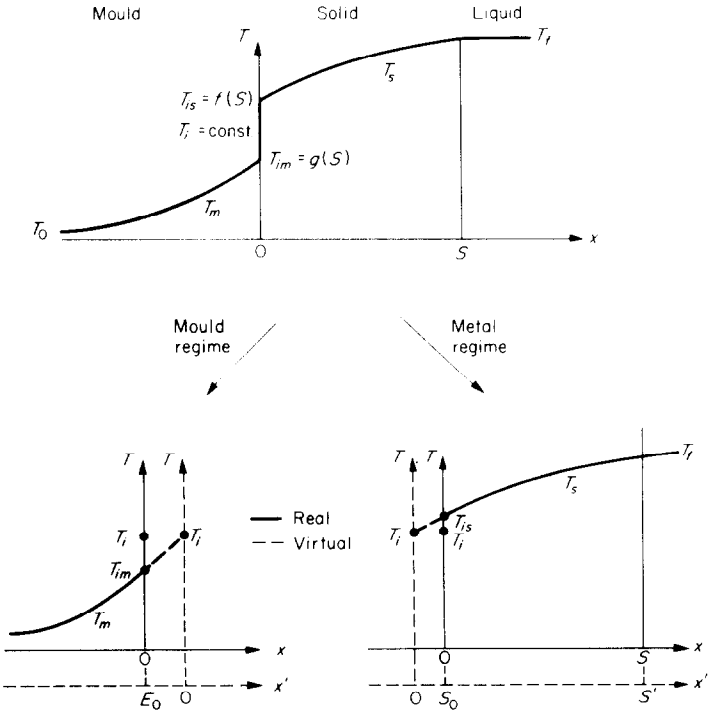

FIG. 1. Division of system into mould and metal components and relationship between real and virtual systems in these regimes.

the situations in which the equation would be expected to apply.

Finally, the predictions of the model when applied to specified cases, which are generalized in terms of heat flow control, will be compared with those produced by application of a standard finite difference numerical technique. This will be done for both chilled and massive uncooled moulds and compared with experimental data.

THEORETICAL

(i) *General model (VAM analysis)*

The model is derived under a set of suppositions similar to those frequently assumed in treatments of the unidirectional solidification problem. Only unidimensional, conductive heat flow is considered, together with Newtonian transfer across the metal/mould interface through a constant heat transfer coefficient, h_i. All material properties are invariants, the freezing interface is macroscopically planar and liquid superheat is negligible. These conditions may all be simulated experimentally and only the last causes significant loss of applicability. Modifications accounting for the effect of superheat may, however, be introduced into both the present and previous models [16, 19]. Freezing from a chill corresponds to a special case of this analysis.

Heat flow is treated in two regimes, separated by a hypothetical plane of constant temperature located in the metal/mould interface. Mould and metal side contributions to the thermal resistance of this interface are now modelled by 'pre-existing' adjuncts of solid material. A two-part coordinate system is set up to describe distance from the plane of constant temperature (on both sides). These two virtual coordinate systems are displaced from the real coordinate origin by the thickness of the virtual adjuncts introduced, as illustrated in Fig. 1. Heat flows in these two regimes are now handled independently, being linked only by equality of heat flux across and temperature (T_i) at the hypothetical plane.

The kinetics of solidification and thermal profile during the process are described by the equations

$$t = \alpha S^2 + \beta S, \tag{1}$$

$$T_s = T_0 + \frac{(T_f - T_0)}{M + \mathrm{erf}(\phi)} \times \left[M + \mathrm{erf}\left(\phi \frac{\beta + 2\alpha x}{\beta + 2\alpha S}\right)\right] \quad 0 \le x \le S, \tag{2a}$$

$$T_m = T_0 + \frac{(T_f - T_0)M}{M + \mathrm{erf}(\phi)} \times \left[1 + \mathrm{erf}\left(\phi \frac{2\alpha N x - \beta}{\beta + 2\alpha S}\right)\right] \quad x \le 0 \tag{2b}$$

where

$$\alpha = \frac{1}{4a_s\phi^2}, \tag{3a}$$

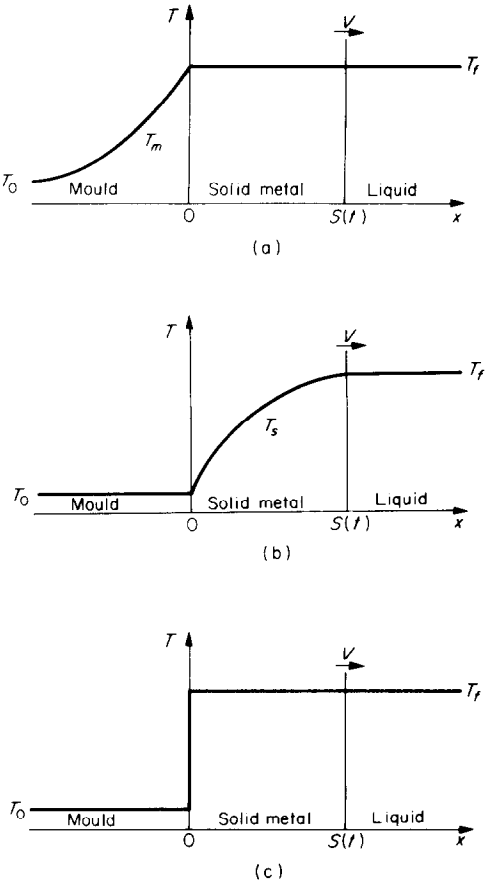

FIG. 2. Thermal profiles during freezing according to the treatments of (a) Chworinoff, (b) Stefan, (c) Flemings.

$$\beta = \frac{Hd_s}{h_i(T_f - T_0)} \quad (= 2S_0\alpha) \tag{3b}$$

and

$$\sqrt{\pi}\,\phi \exp(\phi^2)[M + \mathrm{erf}(\phi)] = \frac{c_s(T_f - T_0)}{H}\left(= \frac{1}{H^*}\right). \tag{4}$$

The meanings of the symbols are given in the Nomenclature. Reference is made to the original papers [1, 2] for an outline of the development of these equations.

(ii) *Limiting cases*

(a) *Insulating mould (Chworinoff treatment).* This case, applicable to moulds of low heat diffusivity, was first described by Chworinoff [20], giving the following:

$$t = \frac{\pi}{4}\left(\frac{Hd_s}{T_f - T_0}\right)^2 \frac{1}{k_m d_m c_m} S^2, \tag{5}$$

$$T_s = T_f, \tag{6a}$$

$$T_m = T_0 + (T_f - T_0)\left\{1 + \mathrm{erf}\left[\frac{x}{2\sqrt{(a_m t)}}\right]\right\}. \tag{6b}$$

The form of the thermal profile represented by equation (6b) is shown in Fig. 2(a).

It may be demonstrated that the same kinetic behaviour is predicted by the VAM analysis: the fact that there is no thermal discontinuity at the metal/mould interface implies that $h_i \rightarrow \infty$, so that $\beta \rightarrow 0$. Furthermore, because ϕ must become small for large M, equation (4) will reduce to

$$\phi = \frac{c_s(T_f - T_0)}{MH\sqrt{\pi}} \tag{7}$$

from which it is easily seen that equations (1) and (5) simplify to the same form. It also follows that equation (2a) will reduce to the correct identity, as the contribution of the two error functions will clearly become negligible as M becomes large and ϕ correspondingly small. Finally, in the limits $\beta \rightarrow 0$ and $M \gg 1$, equation (2b) clearly reduces to

$$T_m = T_0 + (T_f - T_0)\left[1 + \operatorname{erf}\left(\frac{xN\phi}{S}\right)\right] \tag{8}$$

and on substitution for S from equation (5) and for $N[\sqrt{(a_s/a_m)}]$, it can be seen that this is equivalent to the Chworinoff prediction.

(b) *Refrigerated mould/perfect contact (Stefan treatment).* This is one of a number of idealized cases in which the mould is assumed a perfect heat sink, so that it remains at temperature T_0. An implication of this condition is that $M \rightarrow 0$. The second requirement for the model to be a good approximation is that $S \gg (k_s/h_i)$. This is often simply stated as $h_i \rightarrow \infty$, requiring excellent interfacial contact, but it may be additionally noted that the description would be expected to become more appropriate as S increases (i.e. later in the process).

The model was first outlined by Stefan [18] and the relevant equations in this case are

$$t = \frac{1}{4a_s\phi^2}S^2 \tag{9}$$

$$T_s = T_0 + \frac{(T_f - T_0)}{\operatorname{erf}(\phi)}\operatorname{erf}\left(\frac{x}{2\sqrt{(a_s t)}}\right) \tag{10a}$$

$$T_m = T_0 \tag{10b}$$

so that the thermal discontinuity at the metal/mould interface is again assumed negligible, as indicated by the thermal profile shown in Fig. 2(b).

It again follows from the condition $h_i \rightarrow \infty$ that the kinetic descriptions of equations (1) and (9) coincide. For the thermal profile in the solid metal, substitution of the conditions $M = 0$ and $\beta = 0$ in equation (2a) leads to

$$T_s = T_0 + \frac{(T_f - T_0)}{\operatorname{erf}(\phi)}\operatorname{erf}\left(\frac{\phi x}{S}\right), \tag{11}$$

which, on substitution for S from equation (9), gives the same result as equation (10a). The mould temperature is easily seen to be T_0 by substitution of $M = 0$ in equation (2b), so that the model also gives complete agreement in this case.

(c) *Refrigerated mould/interface dominated (Flemings treatment).* This treatment may be regarded as a complementary limiting case to that of Stefan. The mould is again assumed a perfect heat sink, but in this case the thermal resistance of the solidifying metal is assumed small compared with that presented by the metal/mould interface. The requirements for this model may thus be expressed as $M \rightarrow 0$ and $S \ll (k_s/h_i)$. The mathematics are extremely simple because the only temperature change in the thermal profile is a discontinuous drop at the interface. The equation describing the kinetics, presented by Flemings [17], among others, is also very elementary.

$$t = \frac{Hd_s}{h_i(T_f - T_0)}S. \tag{12}$$

The thermal profile has the simple form shown in Fig. 2 (c).

The appropriate form of equation (1) is easily obtained by comparing the contributions of the two terms on the RHS:

$$\frac{\alpha S^2}{\beta S} = \frac{c_s(T_f - T_0)}{4H\phi^2}\left(\frac{h_i S}{k_s}\right). \tag{13}$$

Because the boundary conditions require that $(h_i S/k_s) \rightarrow 0$, it is clear that equation (1) reduces to the same form as equation (12). Similarly, on substituting $(\alpha S/\beta) \rightarrow 0$ (and knowing that the maximum value of x is S) and $M \rightarrow 0$ in equation (2a), this simplifies to the identity $T_s = T_f$. Finally, equation (2b) must reduce to $T_m = T_0$ because of the condition $M \rightarrow 0$.

(iii) *Classical kinetic equation*

A number of equations relating thickness solidified (S) to time (t) have been found to give reasonable agreement with experiment under different circumstances. For example, a simple parabolic relationship holds for casting in insulating moulds and this conforms to the Chworinoff predictions. Similarly, a direct linear relationship is occasionally observed with rapid heat extraction situations, as expected with limiting case (c), although this is not very common with conventional casting set-ups.

However, an equation which is quite frequently found to give a good agreement with experimental behaviour for casting in conducting moulds [21–26] is

$$S = A\sqrt{t} - B, \tag{14}$$

so that the behaviour is predominantly parabolic, but with an apparent 'incubation time' before appreciable solidification takes place – which seems to be finite even for very low superheat. This equation has not been derived analytically from an exact mathematical description, but it will be shown to correspond to behaviour expected from the present model under specified boundary conditions. (Similar inferences may be made from some approximate analytical treatments, such as that of Adams [5].)

It is actually simple to understand qualitatively the kinetic behaviour conforming approximately to the equation in question. Figure 3 illustrates how the initial linear regime (1) gives rise to the observed

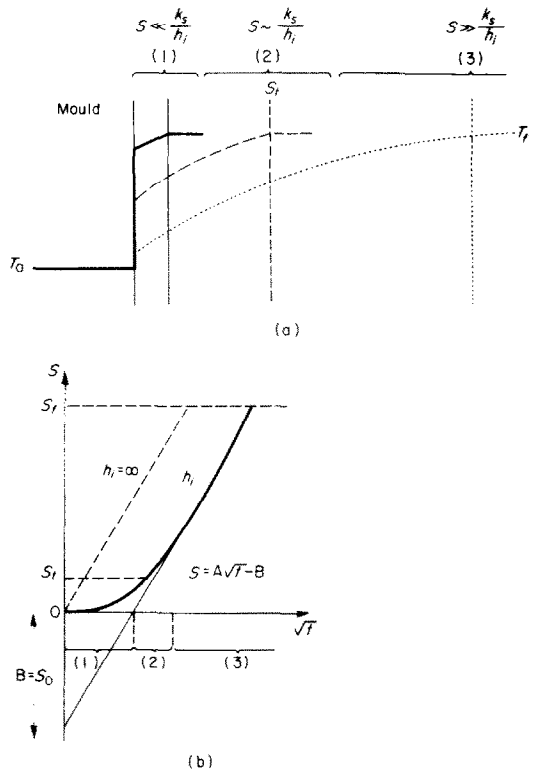

FIG. 3. (a) Successive thermal profiles and (b) $S/\sqrt{t}$ plot for freezing in chill moulds with significant interfacial resistance, showing how the increasing contribution of the thermal resistance of solidified metal results in the initial linear regime (1) giving way to parabolic behaviour (3) through a transition range (2).

displacement (represented by a finite value of B) of the plot for the parabolic regime (3) when compared with the $h_i = \infty$ case. These two regimes are separated by a transition range, the position of which is approximately defined by $S_t (\sim k_s/h_i)$. It might be argued that two conditions are required for the equation to be an appropriate description: that $h_i \ll \infty$—so that the Stefan treatment is inapplicable—and that $S_t \ll S_f$ (the total thickness to be solidified)—so that the disagreement at the start is not significant. These conditions may now be investigated via the present model.

Clearly equation (1) may be rewritten

$$S = \frac{-\beta \pm \sqrt{(\beta^2 + 4\alpha t)}}{2\alpha} \tag{15}$$

from which, taking the meaningful root,

$$S = \sqrt{\left(\frac{t}{\alpha} + S_0^2\right)} - S_0, \tag{16}$$

which is clearly of the form in question when

$$\frac{t}{\alpha} \gg S_0^2, \tag{17a}$$

or, putting this condition in a form which illustrates the requirement in terms of h_i

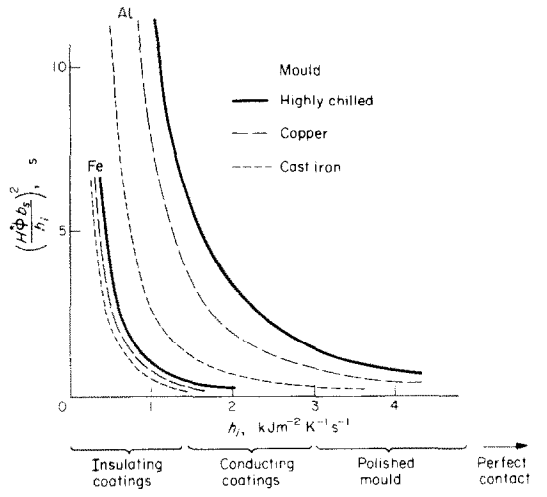

FIG. 4. Variation of the function $(H^*\phi b_s/h_i)^2$ (which controls the duration of the period before the kinetic plot coincides with the equations $S = A\sqrt{t} - B$) with interfacial heat-transfer coefficient for solidification of Fe and Al in different types of conducting moulds.

$$\sqrt{t} \gg \frac{H^*\phi b_s}{h_i} \tag{17b}$$

where the heat diffusivity of the solid metal is now designated b_s. The RHS of this inequality represents the datum point on the $\sqrt{t}$ axis approximately corresponding to the transition region and the equation

$$S = \sqrt{t}\frac{1}{\sqrt{\alpha}} - S_0 \tag{18}$$

is operative for $\sqrt{t}$ values which are appreciably larger than this.

Furthermore, the model can also be used to investigate the $S_t \ll S_f$ condition. Assuming equation (18) to be a good approximation and substituting for

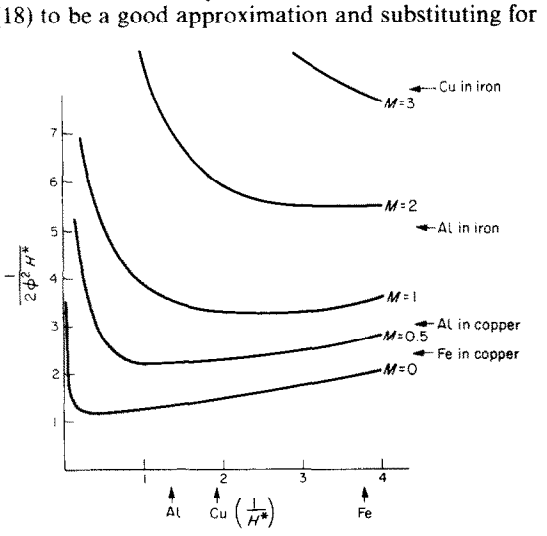

FIG. 5. Dependence of the function $(1/2\,\phi^2 H^*)$, controlling the deviation of the kinetic plot from the equations $S = A\sqrt{t} - B$ during the transition period, on metal/mould parameters $(1/H^*)$ and M.

S_0 from the mathematics of the model [1,2] this condition may be written

$$\sqrt{t_f} \gg \frac{H^* \phi b_s}{h_i}\left(1 + \frac{1}{2\phi^2 H^*}\right) \quad (19)$$

where t_f is the total freezing time. The RHS of this inequality defines a second datum point on the $\sqrt{t}$ axis: if this is considerably greater than that corresponding to the transition region, then a further condition is imposed requiring long t_f (i.e. large S_f) for the equation to be appropriate.

In summary, the RHS of inequality (17b) controls the duration of the initial non-agreement period, and these times are shown as a function of h_i for solidification of steel and aluminium (representative of the common non-ferrous metals) in Fig. 4. Finally, the parameter $(1/2\phi^2 H^*)$, a large value of which will impose the further requirement of a long t_f, is shown as a function of $1/H^*$ and M in Fig. 5. The use of such figures in furthering understanding of the observed kinetics in specified cases is amplified in the discussion section.

(iv) *Generalized cases*

Theoretical examination of generalized cases was undertaken by utilizing a fairly simple computer program describing unidirectional solidification with a planar freezing interface. This was applied to solidification of (a) aluminium from a chill and (b) lead from a massive mould. In the next section these predictions are compared with those from the VAM model and with experimental data.

The computer model used is based on heat transfer between incremental volume elements of finite extent. A number of workers [9–13], [27–30] have outlined applications of the finite difference method to the phase change heat-transfer problem and there have also been studies of programming aspects, including optimization of input/output format [31].

The algorithm to be applied for conduction between successive pairs of elements in metal and mould may be written

$$T_j(t+\delta t) = T_j(t) + \frac{a\delta t}{(\delta x)^2}[T_{j+1}(t) + T_{j-1}(t) - 2T_j(t)] \quad (20)$$

where a is the thermal diffusivity and the step increments δt and δx must be chosen such that

$$\frac{a\delta t}{(\delta x)^2} \leq 0.5. \quad (21)$$

For the solidification problem, the effect of latent heat evolution must be taken into account. This is done by artificially maintaining the Ith element (in solid at solid/liquid interface) at a temperature T_f and accumulating the heat content effectively discounted by doing this: interface advance is only permitted when this discounted heat content reaches the level necessary to freeze the next element.

A heat flux balance at the metal/mould interface employing h_i allows the effect of imperfect contact to be simulated and leads to the algorithms

$$T_{is}(t+\delta t) = T_{is}(t) - 2\left(\frac{a_s \delta t}{\delta x^2}\right)\left\{\frac{\delta x h_i}{k_s}[T_{is}(t) - T_{im}(t)] - [T_{1s}(t) - T_{is}(t)]\right\} \quad (22a)$$

$$T_{im}(t+\delta t) = T_{im}(t) + 2\left(\frac{a_m \delta t}{\delta x^2}\right)\left\{\frac{\delta x h_i}{k_m}[T_{is}(t) - T_{im}(t)] - [T_{im}(t) - T_{1m}(t)]\right\}. \quad (22b)$$

A similar expression may be applied at the outer surface of the mould if heat is being lost to the surroundings.

EXPERIMENTAL AND RESULTS

The generalized cases were examined experimentally by freezing lead against (a) a water-cooled chill and (b) a massive uncooled steel mould. The experimental set-ups used to simulate these two situations have been described [1,2]. In the case of the chill mould, the thickness solidifies (S) was measured at suitable time intervals using a precision dipstick arrangement. For application to massive moulds, however, the position of the freezing front was monitored via the output of a bank of fine thermocouples, accurately located with respect to the metal/mould interface.

Experiments were carried out under two sets of thermal contact conditions at the metal/mould interface, corresponding to the heat-extracting surface being (a) polished and (b) coated with a thin ($\sim 100\ \mu$m) layer of insulating alumina (applied with a spray gun). The appropriate values of h_i for each case were found by suitable manipulation of the kinetic data (see references [1,2]): if a graph is plotted of t/S against S, a straight line is expected, with an intercept on the t/S axis (β) which is related to h_i via properties of the metal

$$h_i = \frac{Hd_s}{(T_f - T_0)\beta}. \quad (23a)$$

In fact, deduction of the value of h_i from the intercept on a graph of t/S against S is quite independent of the actual or assumed kinetic behaviour. This is because, in the limit $S \to 0$, heat flow, and thus the kinetics of interface advance, must in all cases be interface dominated (for the zero superheat case): for this regime, therefore, the Flemings analysis [19] (see (ii) (c) of the Theoretical section) is applicable so that

$$\frac{t}{S} = \frac{Hd_s}{h_i(T_f - T_0)} (= \beta). \quad (23b)$$

Clearly the value of h_i obtained in this way will be that operative at the beginning of freezing. However, linearity of the plot of t/S against S indicates that h_i is remaining constant during solidification.

The following data are presented in dimensionless form, using the metal and mould properties given in Appendix I. By presenting the data in this way, measurements made with different thermal contact

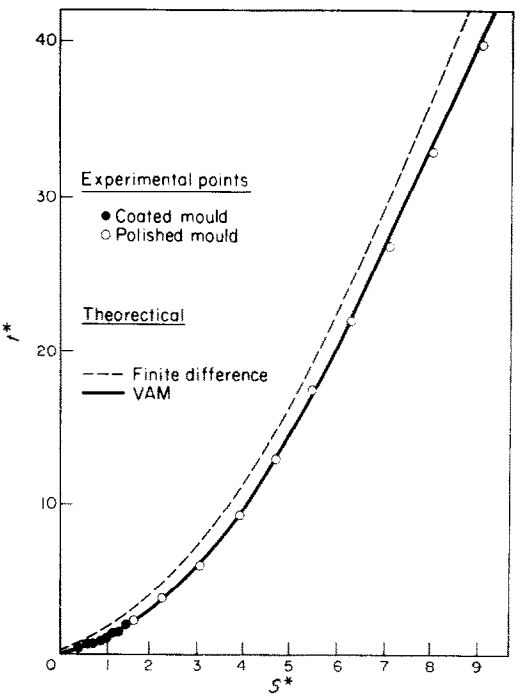

FIG. 6. The kinetics of the freezing of lead from a planar chill. Comparison of the predictions from the VAM analysis with those from a numerical technique. Also shown are experimental data from dipstick measurements.

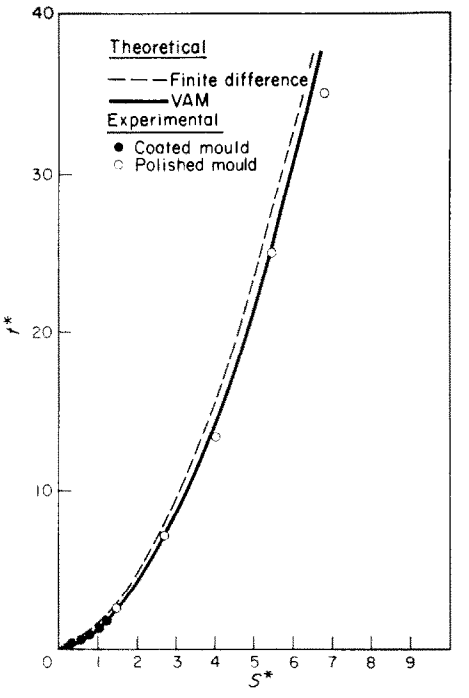

FIG. 8. The kinetics of the freezing of lead from a massive mould (thickness = 100 mm) of low alloy steel. The experimental data were obtained from thermocouple readings.

conditions (coated and polished mould surfaces) can be included on the same graph.

(a) *Chill mould*

Figure 6 shows the datum points obtained from dipstick readings observed at regular time intervals during solidification of lead, converted to dimensionless form by employing the appropriate value of h_i (deduced in the manner outlined above). These are compared with the curves representing the predictions of the VAM model and of the finite difference technique. It can be seen that both predictions are close to the experimental curve.

A further comparison between theory and experiment was made by examining the change in temperature of the solid metal with time. The output from a thermocouple located at the metal/mould interface was combined with the corresponding t/S plot to give the dependence of T_{is} on S. These parameters were then reduced to dimensionless form to give the plots shown in Fig. 7, which refers to a polished mould surface. Agreement is again seen to be quite good.

(b) *Massive mould*

Figure 8 is a kinetic plot for lead freezing against a steel mould of effectively infinite thickness (see [1] for investigation of effectively finite thicknesses). It is again clear that, using the same values for h_i the prediction of the VAM treatment coincides closely with that of the numerical technique as well as giving good agreement with experiment.

Finally, Fig. 9 gives data representing the thermal history of the metal surface for the coated interface

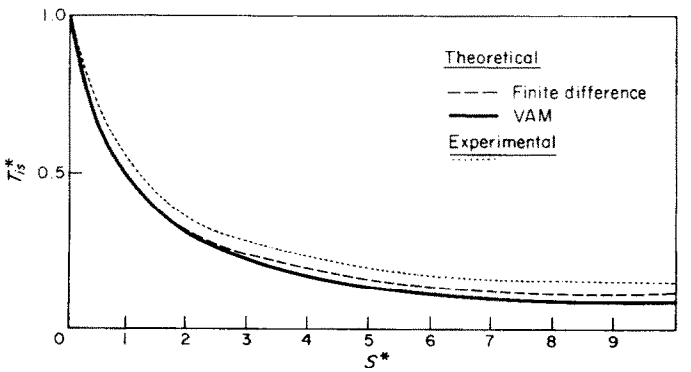

FIG. 7. Thermal history of the metal side of the metal/mould interface for lead freezing from a chill with a polished surface $h_i = 4.2\,\mathrm{kJm^{-2}\,s^{-1}\,K^{-1}}$. The experimental curve was obtained by combining the output from a thermocouple located at the interface with experimental t/S data.

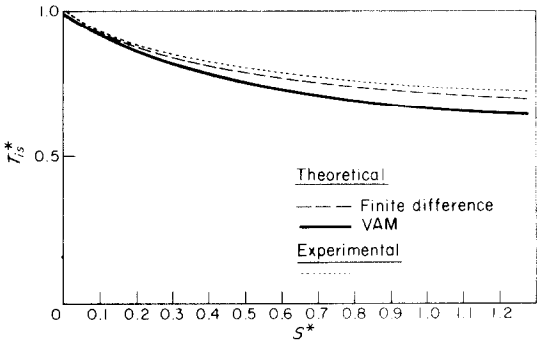

FIG. 9. Comparison between theory and experiment for the thermal history of the metal side of the metal/mould interface for lead solidifying against a massive steel mould with a coated surface. ($h_i = 0.75\,\mathrm{kJm^{-2}\,s^{-1}\,K^{-1}}$).

case, the temperature being plotted against thickness solidified by combination with the appropriate t/S curve. Theoretical predictions are also shown and it can again be seen that the three curves are in good general agreement.

It may be noted parenthetically that the thermal measurements represented in Figs. 7 and 9 may be slightly high due to the difficulty in locating the thermocouple junction exactly at the interface: in practice, it was probably recording the temperature a short distance into the body of the metal. More extensive thermal data are included in the earlier paper [1].

DISCUSSION

It is clear that the thermal and kinetic description of the new model reduces to the expected form in all three of the presented limiting cases. This apparently simple requirement is actually not met by any other exact model. For example, the Schwarz treatment [14], representing the most general previous description, cannot be used to deal with the high interface resistance case.

The utility of the model has been examined by its use to explore the conditions under which the empirical equation $S = A\sqrt{t} - B$ would be expected to be an appropriate discription of the kinetics of solidification. Applying this analysis to individual cases, it can be seen from Fig. 4 that, for given mould and interfacial contact, the actual kinetics will be expected to coincide with the prediction earlier in the process for ferrous casting than for most non-ferrous cases. This observation is particularly apposite for the range of values of h_i ($\sim$0.5–2 $\mathrm{kJm^{-2}\,K^{-1}\,s^{-1}}$) to be expected in practice with various types of mould coating, which is sufficiently low to make the Stefan treatment inaccurate.

A further characteristic which emerges from the analysis concerns the total thickness which must be solidified in order to give the data plot as a whole the general appearance of coincidence with the above equation. Examination of Figs. 4 and 5 will allow this to be investigated for any given case. For example, aluminium solidified in a cast iron mould with $h_i \sim 2\,\mathrm{kJm^{-2}\,K^{-1}\,s^{-1}}$ would give a transition point between linear and parabolic behaviour as early as $\sim$1 s, but inspection of Fig. 5 reveals a value of the function $(1/2\phi^2 H^*)$ of around six so that, from inequality (19), the total solidification time will have to be considerably greater than about 50 times this period. This actually indicates that an appreciable thickness solidifies during the transition period (although this is relatively short), giving a more marked deviation from the prediction of the equation; S_f must therefore be increased to make this less noticeable.

In fact, it also emerges from this aspect that the equation $S = A\sqrt{t} - B$ would be expected to be highly appropriate for all ferrous casting in coated metal moulds, which ties in with experimental observation [21–26]. For non-ferrous metals, it is generally expected to be a much poorer description, although this may not be so for some combinations of h_i, S_f and metal/mould properties. Individual cases can be examined via the present analysis. It is in any event clear that use of the VAM model, which is both relatively simple to employ and of complete generality, is much to be preferred to empirical descriptions, the applications of which are often both restricted and difficult to define in exact terms.

The generality of application of the model has been confirmed by comparison with the predictions of a finite difference numerical model. The most probable area of utility of the VAM approach would be in providing a rapid and simple description of the kinetics of freezing for cases of mixed thermal control (thermal resistances of metal and/or mould and of metal/mould interface of significance). The comparisons presented here have confirmed that a degree of confidence may be placed in the predictions of the model under these circumstances. While it is clear that numerical treatment is indispensible when there are complications such as effectively finite mould thickness, appreciable mushy zone length, markedly non-linear heat flow, continuous variation in h_i etc., it does seem likely that the VAM model will find useful applications in providing, simply and rapidly, a reliable description of a number of real casting situations—an achievement notably lacking with previously-developed exact analyses.

CONCLUSIONS

A new analytical model describing unidirectional solidification of metals freezing with a planar solid–liquid interface has been examined in limiting and generalized cases. It was first confirmed that the description reduced to the correct forms under certain simplifying boundary conditions corresponding to previous exact analyses. The model was then utilized to examine a more complex limiting case—that corresponding to situations in which the empirical equation $S = A\sqrt{t} - B$ gives a good description of the kinetics of freezing. This examination shed light on the kinetic behaviour in such cases and facilitated their

incorporation into an overall rationale. Finally, generalized cases were examined by selecting specific metal/mould/surface condition combinations and assessing the performance of the model against numerical predictions (and experimental data). The descriptions afforded by the model were found to be satisfactory in all cases.

It is clear that the development of an exact analytical model capable of describing the case of heat flow controlled by the thermal resistances of metal, mould and metal/mould interface represents a significant advance. In this paper, it is confirmed that the kinetic and thermal description provided by this model conforms well to experimental conditions under certain imposed constraints and agrees with predictions obtained by numerical mathematical techniques. Exact analyses have hitherto been of very limited use due to their inability to take account of imperfect thermal contact at the metal/mould interface (except when interface resistance dominates those of metal and mould). The present model, however, should be of considerable use in describing real situations.

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APPENDIX I

Metal Properties

Metal	T_f (K)	T_0 (K)	H $\left(\frac{kJ}{kg}\right)$	k_s $\left(\frac{J}{msK}\right)$	c_s $\left(\frac{kJ}{kgK}\right)$	d_s $\left(\frac{mg}{m^3}\right)$
Lead	600	—	25	31	0.138	11.1
Steel (En27)	—	300	—	33	0.486	7.9

EVALUATION D'UN NOUVEAU MODELE DU TRANSFERT THERMIQUE PENDANT LA SOLIDIFICATION UNIDIRECTIONNELLE DES METAUX

Résumé — On évalue un modèle mathématique exact récemment développé par les auteurs [1, 2] pour traiter le problème généralisé de la solidification, uniquement limité par un coefficient de transfert thermique interfacial supposé invariant. La technique suppose l'expédient mathématique de représentation des composantes de la résistance thermique interfaciale par des couches virtuelles de métal solide et/ou de moule et elle est décrite comme méthode de l'adjoint virtuel (VAM). On montre que la description cinétique et thermique prédite par le modèle se réduit à ce qui est attendu dans trois cas simples limites antérieurement étudiés dans une analyse exacte. On montre aussi que l'équation simple $S = A\sqrt{t} - B$, trouvée empiriquement appropriée dans quelques cas, peut être dérivée du modèle sous certaines conditions aux limites. Ceci est appliquée à des situations de forgeage et une analyse est développée pour expliquer les observations expérimentales. Le modèle est ensuite appliqué à des cas de commande thermique mixte. On fait des comparaisons entre les précisions du modèle et les résultats du calcul numériques pour des situations dans lesquelles les résistances thermiques du métal/interface sont significatives. Pour cela on utilise les mesures des coefficients de transfert thermique et la comparaison est étendue pour couvrir les données cinétiques et thermiques. Le modèle est montré être excellent.

BEWERTUNG EINES NEUEN MODELLS FÜR DEN WÄRMESTROM WÄHREND DER EINDIMENSIONALEN ERSTARRUNG VON METALLEN

Zusammenfassung—Diese Studie ist die Anwendung eines exakten mathematischen Modells, das vor kurzem von den Verfassern [1–2] entwickelt wurde, um das allgemeine Erstarrungsproblem zu behandeln, mit der einzigen einschränkenden Bedingung, daß der Wärmeübergangskoeffizient zwischen den Grenzflächen konstant ist. Die Methode enthält das mathematische Hilfsmittel der Darstellung von Anteilen des thermischen Widerstandes der Grenzschicht durch virtuelle Schichten von festem und/oder geschmolzenem Metall und wird beschrieben als die Methode der Virtuellen Zusätze (VAM). Es wird gezeigt, daß die vom Modell hergeleitete kinematische und thermische Beschreibung in drei einfachen Grenzfällen, die vorher einer genauen Analyse unterzogen wurden, auf das Erwartete führt. Es wird außerdem gezeigt, daß die einfache kinetische Gleichung $S = A\sqrt{t} - B$, die empirische in einigen Fällen als geeignet erscheint, unter bestimmten Randbedingungen aus dem Modell abgeleitet werden kann. Diese werden für bestimmte Situationen des Formgießens geprüft und angewendet, wodurch anscheinend gewisse experimentelle Beobachtungen erklärt werden können. Letztlich wird das Modell in allgemeinen Fällen gemischter Temperaturführung geprüft. Die Berechnungen nach dem Modell und Ergebnisse von numerischen Rechnungen werden verglichen, bei denen der thermische Widerstand von (a) Metall und Grenzfläche und (b) Metall, Grenzfläche und Form von Bedeutung sind. Dieses geschieht unter Verwendung gemessener Werte von Wärmeübergangskoeffizienten, wobei der Vergleich auf experimentelle kinetische und thermische Daten ausgedehnt wird. Das Modell erweist sich als ausgezeichnet.

ОЦЕНКА НОВОЙ МОДЕЛИ ТЕПЛООБМЕНА ПРИ НАПРАВЛЕННОМ ЗАТВЕРДЕВАНИИ МЕТАЛЛОВ

Аннотация — Исследование проведено с целью проверки точной математической модели, предложенной авторами ранее для решения обобщенной задачи затвердевания в случае, когда коэффициент теплопереноса на границе раздела фаз является постоянным. Метод основывается на послойном (для твердого металла и/или прессформы) представлении компонент межфазового теплового сопротивления и описывается как метод виртуальных дополнений. Показано, что с помощью данной модели кинетическое и тепловое описание можно свести к трем простым предельным случаям, точный анализ которых был выполнен ранее. Показано, что при определенных граничных условиях из модели можно вывести простое кинетическое уравнение $S = A\sqrt{t} - B$, являющееся справедливым для ряда случаев. Проведено исследование этих граничных условий, которые применены к специальным случаям литья. Проведена обработка экспериментальных результатов. И наконец, модель проверена на общих случаях смешанного теплового контроля. Проведено сравнение результатов, полученных с помощью модели с результатами численных расчетов, когда существенными являются тепловое сопротивление (а) металла и границы раздела и (б) металла, границы раздела и пресс-формы. С этой целью измерялись коэффициенты теплообмена и использовались экспериментальные данные по кинетике процесса и теплообмену. Показана пригодность модели для рассматриваемого класса задач.